3.6 : Fundamental Solution Set

Consider the following IVP:

$$p(t)y'' + q(t)y' + r(t)y = 0 \qquad (1)$$

$$y(t_0) = y_0 \;, \quad y'(t_0) = y_0'$$

Suppose that y_1, y_2 are two solutions to (1). Then, by the Principle of Superposition,

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is also a solution to (1).

Note that $y'(t) = C_1 y_1'(t) + C_2 y_2'(t)$

Using initial conditions in (1):

$$y_0 = y(t_0) = C_1 y_1(t_0) + C_2 y_2(t_0) \qquad (2.1)$$

$$y_0' = y'(t_0) = C_1 y_1'(t_0) + C_2 y_2'(t_0) \qquad (2.2)$$

We will first solve C_2 in terms of C_1 in (2.1) (skip Cramer's rule and matrix algebra), then we will substitute C_2 solved from (2.1) into (2.2) to solve C_1 in terms of known quantities. Finally, we will use either (2.1) or (2.2) to solve C_2.

Step 1: From (2.1):

$$C_2 = \frac{y_0 - C_1 y_1(t_0)}{y_2(t_0)} \qquad (*)$$

Step 2: Substitute $(*)$ into (2.2):

$$y_0' = C_1 y_1'(t_0) + \frac{y_0 - C_1 y_1(t_0)}{y_2(t_0)} y_2'(t_0)$$

$$= C_1 y_1'(t_0) - C_1 \frac{y_1(t_0) y_2'(t_0)}{y_2(t_0)} + \frac{y_0 \cdot y_2'(t_0)}{y_2(t_0)}$$

$$= C_1 \left[y_1'(t_0) - \frac{y_1(t_0)\, y_2'(t_0)}{y_2(t_0)} \right] + \frac{y_0 \cdot y_2'(t_0)}{y_2(t_0)}$$

$$= C_1 \frac{y_1'(t_0)\, y_2(t_0) - y_1(t_0)\, y_2'(t_0)}{y_2(t_0)} + \frac{y_0 \cdot y_2'(t_0)}{y_2(t_0)}$$

$$\Rightarrow y_0' - \frac{y_0 \cdot y_2'(t_0)}{y_2(t_0)} = C_1 \frac{y_1'(t_0)\, y_2(t_0) - y_1(t_0)\, y_2'(t_0)}{y_2(t_0)}$$

$$\Rightarrow \frac{y_0'\, y_2(t_0) - y_0\, y_2'(t_0)}{y_2(t_0)} = C_1 \frac{y_1'(t_0)\, y_2(t_0) - y_1(t_0)\, y_2'(t_0)}{y_2(t_0)}$$

$$\Rightarrow y_0'\, y_2(t_0) - y_0\, y_2'(t_0) = C_1 \left(y_1'(t_0)\, y_2(t_0) - y_1(t_0)\, y_2'(t_0) \right)$$

$$\Rightarrow C_1 = \frac{y_0'\, y_2(t_0) - y_0\, y_2'(t_0)}{y_1'(t_0)\, y_2(t_0) - y_1(t_0)\, y_2'(t_0)}$$

$$= \frac{y_0\, y_2'(t_0) - y_0'\, y_2(t_0)}{y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0)}$$

$$= \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

(Cramer's rule was correct!)

(3)

Note : $\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = W(y_1(t), y_2(t))(t_0)$

(Wronskian of y_1, y_2)

From above it is clear why we need the Wronskian to be nonzero.

Substituting (3.1) into (2.1) :

$$y_0 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)} y_1(t_0) + C_2 y_2(t_0)$$

$$\Rightarrow \quad C_2 y_2(t_0) = y_0 - \frac{\left(y_0 y_2'(t_0) - y_0' y_2(t_0)\right) y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

$$\Rightarrow C_2 y_2(t_0)$$

$$= \frac{y_0\left(y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)\right) - \left(y_0 y_2'(t_0) - y_0' y_2(t_0)\right) y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

$$C_2 = \frac{\cancel{y_0 y_1(t_0) y_2'(t_0)} - y_0 y_1'(t_0) y_2(t) - \cancel{y_0 y_2'(t_0) y_1(t_0)} + y_0' y_2(t_0) y_1(t_0)}{y_2(t_0)\left(y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)\right)}$$

$$= \frac{y_0' \, y_1(t_0) - y_0 \, y_1'(t_0)}{y_1(t_0) \, y_2'(t_0) - y_1'(t_0) \, y_2(t_0)}$$

$$= \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$ (Cramer's rule works again!)

It might be beneficial for you to do the algebraic calculation above (for general case) once, then keep the results of Cramer's rule in mind for simplicity)